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LOWER BOUNDS FOR THE RANK AND LOCATION OF THE EIGENVALUES OF A MATRIX

by

Ky Fan and A. J. Hoffman



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This paper contains several remarks devoted to the problems mentioned in the title. The results of §1 apply to arbitrary $n \times n$ matrices with complex coefficients, those of §2 apply to normal matrices only.

1

In this section we consider the following problems concerning an arbitrary $n \times n$ matrix $A = (a_{ij})$ with complex coefficients:

Problem 1: Find lower bounds for the rank of A that can be calculated in a simple manner from the coefficients a_{ij} .

Problem 2: Find n non-negative numbers ρ_1, \dots, ρ_n such that every eigenvalue of A lies in one or more of the n circular disks

$$|\lambda - a_{ii}| \leq \rho_i \quad (i = 1, \dots, n).$$

Most of the results in the literature relevant to Problem 1 are sufficient conditions for a matrix to be non-singular,

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such as Hadamard's theorem that $|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|$ ($1 \leq i \leq n$) implies that A is non-singular. As mentioned in [8], the application of this theorem to $A - \lambda I$ implies the theorem of S. Geršgorin [3] that setting $\rho_i = \sum_{j=1, j \neq i}^n |a_{ij}|$ ($1 \leq i \leq n$) is a solution to Problem 2 (See [8] and [10] for an extensive bibliography on solutions to Problem 2). A generalization by A. Ostrowski [5] of the theorem of Hadamard-Geršgorin is the following: If p and q are positive, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha_1, \dots, \alpha_n$ are n positive numbers such that

$$\sum_{i=1}^n \frac{1}{1 + \alpha_i} \leq 1, \quad (1.1)$$

then $|a_{ii}| > \alpha_i^{1/q} \left(\sum_{j=1, j \neq i}^n |a_{ij}|^p \right)^{1/p}$ ($1 \leq i \leq n$) implies that

A is non-singular; or, equivalently, setting

$$\rho_i = \alpha_i^{1/q} \left(\sum_{j=1, j \neq i}^n |a_{ij}|^p \right)^{1/p} \quad (1 \leq i \leq n) \quad \text{is a solution to}$$

Problem 2. (Hadamard-Geršgorin's theorem is the limiting case $p = 1$ of this result). An improvement of Hadamard-Gersgorin's theorem is a theorem of Taussky [9] and P. Stein [6] which asserts that if λ is an eigenvalue with s linearly independent eigenvectors corresponding to it, then λ lies in at least s of the Hadamard-Geršgorin disks. Theorem D of [6] points out that consideration of $\lambda = 0$ implies a theorem about the rank. The following theorem, formulated in terms of Problem 1, is a slight extension of this result:

THEOREM 1.1. Let $A = (a_{ij})$ be an $n \times n$ matrix. For any two integers i, m ($1 \leq i \leq n$, $2 \leq m \leq n$), let $b_i(m)$ be the

maximum sum of the moduli of $m-1$ distinct off-diagonal elements of the i -th row of A , i.e.

$$b_i(m) = \max_{j_1, \dots, j_{m-1} \neq i} (|a_{ij_1}| + \dots + |a_{ij_{m-1}}|). \quad (1.2)$$

If, for some m ,

$$|a_{ii}| > b_i(m) \quad (1.3)$$

holds for at least m distinct indices i , then the rank r of A is at least m .

Proof: Consider the $m \times m$ submatrix B obtained from A by deleting the rows with indices for which (1.3) is not satisfied, and the corresponding columns. By (1.2) and (1.3), Hadamard's theorem applies to B , hence B is non-singular, which implies $r \geq m$.

If λ is an eigenvalue of A with s linearly independent eigenvectors corresponding to it, then the rank of $A - \lambda I$ is $n-s$, and the theorem of Taussky and Stein follows at once from Theorem 1.1. It is also easy to see how Theorem 1.1 and the theorem of Taussky and Stein can be generalized by using Ostrowski's theorem quoted above. They may also be extended by using another theorem of Ostrowski [4] which asserts that if ρ_i and σ_i are the radii of the Hadamard-Geršgorin disks for A and A^* respectively, and $0 \leq \alpha \leq 1$, then $\tau_i = \rho_i^\alpha \sigma_i^{1-\alpha}$ ($1 \leq i \leq n$) also is a solution of Problem 2.

THEOREM 1.2. For any $n \times n$ matrix $A = (a_{ij})$ of rank r ,

we have

$$\sum_{i=1}^n \frac{|a_{ii}|^2}{\sum_{j=1}^n |a_{ij}|^2} \leq r. \quad (1.4)$$

(Whenever $\frac{0}{0}$ occurs on the left-hand side, we agree to put $\frac{0}{0} = 0$.)

Proof: Let a_i denote the i -th row vector of A and e_i the i -th unit vector. Both sides of (1.4) remain unchanged, if we multiply any row of A by a number $\neq 0$. Hence we may assume that for each i , $\|a_i\|^2 = \sum_{j=1}^n |a_{ij}|^2 = 1$ or 0 . We have to prove, under this assumption, that $\sum_{i=1}^n |(a_i, e_i)|^2 \leq r$.

As A is of rank r , we can find n orthonormal vectors x_1, x_2, \dots, x_n such that

$$(a_i, x_j) = 0 \quad \text{for} \quad \begin{cases} 1 \leq i \leq n \\ r+1 \leq j \leq n. \end{cases}$$

For each i , we have

$$(a_i, e_i) = \sum_{j=1}^n (a_i, x_j) (\overline{e_i, x_j}) = \sum_{j=1}^r (a_i, x_j) (\overline{e_i, x_j}),$$

and therefore

$$|(a_i, e_i)|^2 \leq \left(\sum_{j=1}^r |(a_i, x_j)|^2 \right) \left(\sum_{j=1}^r |(e_i, x_j)|^2 \right).$$

Since $\sum_{j=1}^r |(a_i, x_j)|^2 = \|a_i\|^2 = 1$ or 0 , we have

$$|(a_i, e_i)|^2 \leq \sum_{j=1}^r |(e_i, x_j)|^2. \quad (1 \leq i \leq n)$$

Consequently

$$\sum_{i=1}^n |(a_i, e_i)|^2 \leq \sum_{j=1}^r \sum_{i=1}^n |(e_i, x_j)|^2 = \sum_{j=1}^r \|x_j\|^2 = r.$$

THEOREM 1.3. For any $n \times n$ matrix $A = (a_{ij})$ of rank r , we have

$$\sum_{i=1}^n \frac{|a_{ii}|}{\sum_{j=1}^n |a_{ij}|} \leq r. \quad (1.5)$$

(We keep our agreement $\frac{0}{0} = 0$).

Proof: For the reason explained at the beginning of the proof of Theorem 1.2, we may assume

$$a_{ii} \geq 0 \quad (1 \leq i \leq n) \quad (1.6)$$

and

$$\sum_{j=1}^n |a_{ij}| = 1 \text{ or } 0. \quad (1 \leq i \leq n) \quad (1.7)$$

It suffices to prove that $\sum_{i=1}^n a_{ii} \leq r$ holds for any matrix $A = (a_{ij})$ with rank r and satisfying (1.6), (1.7).

By Hadamard-Geršgorin's theorem mentioned above, (1.7) implies that all eigenvalues of A have modulus ≤ 1 . Therefore $\sum_{i=1}^n a_{ii}$ is not larger than the number of non-zero eigenvalues of A . But the number of non-zero eigenvalues of A is $\leq r^{(*)}$, hence $\sum_{i=1}^n a_{ii} \leq r$.

(*) Let T be a non-singular matrix such that $T^{-1}AT$ is triangular. Then the rank of A is same as the rank of $T^{-1}AT$ and the latter is obviously \geq the number of non-zero eigenvalues of A .

Theorem 1.3 may be generalized as follows:

THEOREM 1.4. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n positive numbers satisfying (1.1). Then for any $n \times n$ matrix $A = (a_{ij})$ of rank r , we have

$$\sum_{i=1}^n \frac{|a_{ii}|}{|a_{ii}| + \alpha_i^{1/q} \left(\sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|^p \right)^{1/p}} < r. \quad (1.8)$$

(We keep our agreement $\frac{0}{0} = 0$.)

Proof: The proof is similar to that of Theorem 1.3. Here it suffices to show that, if a matrix $A = (a_{ij})$ satisfies (1.6) and

$$|a_{ii}| + \alpha_i^{1/q} \left(\sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|^p \right)^{1/p} = 1 \text{ or } 0, \quad (1 \leq i \leq n) \quad (1.9)$$

then $\sum_{i=1}^n a_{ii} \leq r$.

According to Ostrowski's theorem [5] mentioned above, (1.9) implies that all eigenvalues of A have modulus ≤ 1 . This fact together with (1.6) implies $\sum_{i=1}^n a_{ii} \leq r$.

Theorem 1.3 may be regarded as the limiting case $p \rightarrow 1$ of Theorem 1.4. Because of its relative simplicity, we state explicitly the other limiting case $p \rightarrow \infty$ of Theorem 1.4:

COROLLARY 1.1. If n positive numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ satisfy inequality (1.1), then for any $n \times n$ matrix $A = (a_{ij})$ of rank r , we have

$$\sum_{i=1}^n \frac{|a_{ii}|}{|a_{ii}| + \alpha_i \max_{\substack{1 \leq j \leq n \\ j \neq i}} |a_{ij}|} \leq r. \quad (1.10)$$

We turn now to results concerning Problem 2.

THEOREM 1.5. Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\alpha > 0$ satisfies

$$\sum_{i=1}^n \frac{(\sum_{j \neq i} |a_{ij}|)^q}{(\sum_{j \neq i} |a_{ij}|^p)^{q/p}} \leq \alpha^q (1 + \alpha^q) \quad (1.11)$$

(where we keep our agreement $\frac{0}{0} = 0$), then every eigenvalue λ of A lies in at least one of the n circular disks:

$$|\lambda - a_{ii}| \leq \alpha \left(\sum_{j \neq i} |a_{ij}|^p \right)^{1/p}. \quad (1 \leq i \leq n) \quad (1.12)$$

Proof: Let λ be an eigenvalue of A and let $\{x_1, x_2, \dots, x_n\}$ be a corresponding eigenvector:

$$\lambda x_i = \sum_{j=1}^n a_{ij} x_j. \quad (1 \leq i \leq n) \quad (1.13)$$

We may assume that

$$\sum_{i=1}^n |x_i|^q = 1. \quad (1.14)$$

Suppose, if possible that

$$|\lambda - a_{ii}| > \alpha \left(\sum_{j \neq i} |a_{ij}|^p \right)^{1/p}. \quad (1 \leq i \leq n) \quad (1.15)$$

Then for any index i such that $x_i \neq 0$, we have by (1.15), (1.13), (1.14):

$$\begin{aligned} \alpha^q \left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p} |x_i|^q &< |\lambda - a_{ii}|^q |x_i|^q = \\ &= \left| \sum_{j \neq i} a_{ij} x_j \right|^q \leq (1 - |x_i|^q) \left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p} \end{aligned}$$

and therefore

$$|x_i|^q < \frac{1}{1 + \alpha^q}. \quad (1.16)$$

This inequality is of course also true when $x_i = 0$. It follows that

$$\left| \sum_{j \neq i} a_{ij} x_j \right| \leq \left(\frac{1}{1 + \alpha^q} \right)^{1/q} \sum_{j \neq i} |a_{ij}|$$

or

$$|\lambda - a_{ii}|^q |x_i|^q \leq \frac{1}{1 + \alpha^q} \left(\sum_{j \neq i} |a_{ij}| \right)^q. \quad (1 \leq i \leq n)$$

Combining (1.15) with the last inequality, we infer that

$$\alpha^q \left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p} |x_i|^q < \frac{1}{1 + \alpha^q} \left(\sum_{j \neq i} |a_{ij}| \right)^q$$

holds for every index i such that $x_i \neq 0$. Hence

$$|x_i|^q < \frac{1}{\alpha^q (1 + \alpha^q)} \cdot \frac{\left(\sum_{j \neq i} |a_{ij}| \right)^q}{\left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p}}$$

holds whenever $x_i \neq 0$. The weaker inequality

$$|x_i|^q \leq \frac{1}{\alpha^q (1 + \alpha^q)} \cdot \frac{\left(\sum_{j \neq i} |a_{ij}| \right)^q}{\left(\sum_{j \neq i} |a_{ij}|^p \right)^{q/p}}$$

is of course satisfied for all i . Thus, summing this inequality over i and using (1.14), we get an inequality contradicting (1.11).

As limiting case $p \rightarrow \infty$ of Theorem 1.5, we have

COROLLARY 1.2. Let $A = (a_{ij})$ be an $n \times n$ matrix. If $\alpha > 0$ is such that

$$\sum_{i=1}^n \frac{\sum_{j \neq i} |a_{ij}|}{\max_{j \neq i} |a_{ij}|} \leq \alpha (1 + \alpha) \quad (1.17)$$

(where we keep our agreement $\frac{0}{0} = 0$), then every eigenvalue of A lies in at least one of the n circular disks

$$|\lambda - a_{ii}| \leq \alpha \max_{j \neq i} |a_{ij}|. \quad (1 \leq i \leq n) \quad (1.18)$$

In the same way that Theorems 1.3, 1.4 are derived from Hadamard-Geršgorin's theorem and Ostrowski's theorem respectively, other lower bounds for the rank can be derived from our Theorem 1.5. In particular, as a consequence of Corollary 1.2, we have

COROLLARY 1.3. Let $A = (a_{ij})$ be an $n \times n$ matrix of rank r . If α is a positive number satisfying (1.17), then

$$\sum_{i=1}^n \frac{|a_{ii}|}{|a_{ii}| + \alpha \max_{j \neq i} |a_{ij}|} \leq r. \quad (1.19)$$

Using theorem 1.2, we may improve the case $p = 2$ of Ostrowski's theorem [5].

THEOREM 1.6. Let λ be an eigenvalue of an $n \times n$ matrix $A = (a_{ij})$ and let s be the number of linearly independent eigenvectors corresponding to λ . If n positive numbers $\beta_1, \beta_2, \dots, \beta_n$ are such that

$$\sum_{i=1}^n \frac{1}{1 + \beta_i} \leq s, \quad (1.20)$$

then λ lies in at least one of the n disks:

$$|\lambda - a_{ii}| \leq \beta_i^{\frac{1}{2}} \left(\sum_{j \neq i} |a_{ij}|^2 \right)^{\frac{1}{2}}. \quad (1 \leq i \leq n) \quad (1.21)$$

Proof: Since s is the number of linearly independent eigenvectors of A corresponding to the eigenvalue λ , the rank of the matrix $A - \lambda I$ is $n-s$. Applying Theorem 1.2 to $A - \lambda I$, we get

$$\sum_{i=1}^n \frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq n-s.$$

As (1.20) may be written

$$n - s \leq \sum_{i=1}^n \frac{\beta_i}{1 + \beta_i},$$

we have

$$\sum_{i=1}^n \frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq \sum_{i=1}^n \frac{\beta_i}{1 + \beta_i}.$$

Therefore at least one index i satisfies

$$\frac{|a_{ii} - \lambda|^2}{|a_{ii} - \lambda|^2 + \sum_{j \neq i} |a_{ij}|^2} \leq \frac{\beta_i}{1 + \beta_i},$$

which is precisely (1.21).

Similarly, from Theorem 1.3, we can derive the following theorem:

THEOREM 1.7. Let λ be an eigenvalue of an $n \times n$ matrix $A = (a_{ij})$ and let s be the number of linearly independent eigenvectors corresponding to λ . If n positive numbers $\beta_1, \beta_2, \dots, \beta_n$ satisfy (1.20), then λ lies in at least one of the n disks:

$$|\lambda - a_{ii}| \leq \beta_i \sum_{j \neq i} |a_{ij}|. \quad (1 \leq i \leq n) \quad (1.22)$$

It is interesting to compare this result with the theorem of Taussky and Stein discussed in connection with Theorem 1.1 above.

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Several inclusion theorems for the eigenvalues of a normal matrix are already known. Among them, recent results are due to L. Collatz [1], H. Wielandt [12], A. G. Walker and J. D. Weston [11]. All these results are concerned with a single eigenvalue. In this § we shall give inclusion theorems concerning k eigenvalues of an $n \times n$ normal matrix, k being any positive integer $\leq n$. Our results are based on the following extremal property of the eigenvalues of normal matrices: Let the eigenvalues λ_i ($1 \leq i \leq n$) of a normal matrix N be so arranged that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. Then for any positive integer $k \leq n$, we have

$$\sum_{i=1}^k |\lambda_i|^2 = \text{Max} \sum_{j=1}^k \|Nx_j\|^2, \quad (2.1)$$

$$\sum_{i=1}^k |\lambda_{n-i+1}|^2 = \text{Min} \sum_{j=1}^k \|Nx_j\|^2, \quad (2.2)$$

where, for both maximum and minimum, x_1, x_2, \dots, x_k runs over all sets of k orthonormal vectors in the unitary n -space. As N^*N is a Hermitian matrix with eigenvalues $|\lambda_i|^2$ ($1 \leq i \leq n$), this follows from a similar property for the eigenvalues of Hermitian matrices ([2], Theorem 1).

THEOREM 2.1. Let N be an $n \times n$ normal matrix with eigenvalues λ_i ($1 \leq i \leq n$). Let x_1, x_2, \dots, x_k be k ($\leq n$) orthonormal vectors and γ be a complex number, δ a non-negative real number such that

$$\sum_{j=1}^k \|(N - \gamma I)x_j\|^2 \leq \delta. \quad (2.3)$$

Then there exist k distinct indices $\nu_1, \nu_2, \dots, \nu_k$ among $1, 2, \dots, n$ such that

$$\sum_{i=1}^k |\lambda_{\nu_i} - \gamma|^2 \leq \delta. \quad (2.4)$$

The theorem remains true, when both signs " \leq " in (2.3), (2.4) are reversed.

Proof: Consider the normal matrix $N - \gamma I$, whose eigenvalues are $\lambda_i - \gamma$ ($1 \leq i \leq n$). If we rearrange the λ_i 's into $\lambda_{\nu_1}, \lambda_{\nu_2}, \dots, \lambda_{\nu_n}$ such that

$$|\lambda_{\nu_1} - \gamma| \leq |\lambda_{\nu_2} - \gamma| \leq \dots \leq |\lambda_{\nu_n} - \gamma|,$$

then applying the minimum property (2.2) to $N - \gamma I$, we obtain

$$\sum_{i=1}^k |\lambda_{\nu_i} - \gamma|^2 \leq \sum_{j=1}^k \|(N - \gamma I)x_j\|^2. \quad (2.5)$$

In view of (2.3), (2.5) implies (2.4).

Similarly one proves the second part of the theorem by applying the maximum property (2.1) to $N - \gamma I$.

Remark: Let α and β be any two complex numbers. Put

$$\gamma = \frac{\alpha + \beta}{2}, \quad \delta = k \left| \frac{\alpha - \beta}{2} \right|^2$$

Then in the theorem just proved, condition (2.3) becomes

$$\operatorname{Re} \sum_{j=1}^k ((N - \alpha I)x_j, (N - \beta I)x_j) \leq 0. \quad (2.6)$$

Using this form (2.6) of condition (2.3), we see that the case $k = 1$ of Theorem 2.1 implies the following result:

COROLLARY 2.1. Let N be an $n \times n$ normal matrix. Let $x = \{x_1, x_2, \dots, x_n\}$ be a vector different from the zero-vector. Let $Nx = \{y_1, y_2, \dots, y_n\}$. If α, β are two complex numbers such that

$$\operatorname{Re} \sum_{i=1}^n (y_i - \alpha x_i)(\overline{y_i - \beta x_i}) \leq 0, \quad (2.7)$$

then N has at least one eigenvalue in the disk

$$\left| \lambda - \frac{\alpha + \beta}{2} \right| \leq \left| \frac{\alpha - \beta}{2} \right|$$

The corollary remains true, when both signs " \leq " are reversed.

This result improves slightly an inclusion theorem given independently by H. Wielandt [12] and Walker-Weston [11]. These authors assume

$$\operatorname{Re} [(y_i - \alpha x_i)(\overline{y_i - \beta x_i})] \leq 0, \quad (1 \leq i \leq n)$$

which is stronger than (2.7). On the other hand, the inclusion theorem of Wielandt-Walker-Weston generalizes an inclusion theorem of L. Collatz [1] concerning real symmetric matrices.

THEOREM 2.2. Let N be an $n \times n$ normal matrix with eigenvalues $\lambda_i (1 \leq i \leq n)$. For any $k (\leq n)$ orthonormal vectors x_1, x_2, \dots, x_k in the unitary n -space, there exist k distinct indices $\nu_1, \nu_2, \dots, \nu_k$ among $1, 2, \dots, n$ such that

$$\sum_{i=1}^k \left| \lambda_{\nu_i} - \frac{1}{k} \sum_{j=1}^k (Nx_j, x_j) \right|^2 \leq \sum_{j=1}^k \|Nx_j\|^2 - \frac{1}{k} \left| \sum_{j=1}^k (Nx_j, x_j) \right|^2. \quad (2.8)$$

There also exist k distinct indices $\nu_1, \nu_2, \dots, \nu_k$ satisfying the reversed inequality of (2.8).

Proof: Let

$$\bar{\gamma} = \frac{1}{k} \sum_{j=1}^k (Nx_j, x_j),$$

$$\delta = \sum_{j=1}^k \|Nx_j\|^2 - \frac{1}{k} \left| \sum_{j=1}^k (Nx_j, x_j) \right|^2.$$

One sees that

$$\sum_{j=1}^k \|(N - \bar{\gamma} I)x_j\|^2 = \delta,$$

which incidentally implies that $\delta \geq 0$.

Hence the first part of Theorem 2.1 implies the existence of ν_1, \dots, ν_k satisfying (2.8), and the second part of Theorem 2.1 implies the existence of ν_1, \dots, ν_k satisfying the reverse of inequality (2.8).

The case $k = 1$ of Theorem 2.2 is a known result obtained independently by Wielandt [12] and Walker-Weston [11].

For its simplicity, we state explicitly the following special case of the case $k = 1$ of Theorem 2.2. Let $N = (a_{ij})$ be an $n \times n$ normal matrix. Let D_i denote the disk $|z - a_{ii}| \leq \left(\sum_{j \neq i} |a_{ij}|^2 \right)^{\frac{1}{2}}$. Then for each $i = 1, 2, \dots, n$, D_i contains at least one eigenvalue of N and the interior of D_i never contains all n eigenvalues of N .

References

- [1] L. Collatz, "Einschliessungssatz für die charakteristischen Zahlen von Matrizen," Math. Zeitschr., 48, (1942), 221-226.
- [2] K. Fan, "On a theorem of Weyl concerning Eigenvalues of linear transformations, I," Proc. Nat. Acad. Sci. USA, 35(1949) 652-655.
- [3] S. Geršgorin, "Über die Abgrenzung der Eigenwerte einer Matrix," Izv. Akad. Nauk S.S.S.R., 7(1931), 749-754.
- [4] A. Ostrowski, "Über das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen," Compositio Math. 9(1951), 209-226.
- [5] A. Ostrowski, "Sur les conditions générales pour la régularité des matrices," Rend. di Matem.e delle sue appl., 10(1951), 156-168.
- [6] P. Stein, "A Note on Bounds of Multiple Characteristic Roots of a Matrix," Journal of Research of the National Bureau of Standards, 48(1952), 59-60.
- [7] O. Taussky-Todd, "Bounds for characteristic roots of matrices," Duke. Math. J., 15(1948), 1043-1044.
- [8] O. Taussky-Todd, "A recurring theorem on determinants," Amer. Math. Monthly, 56(1949), 672-676.
- [9] O. Taussky-Todd, "Bounds for characteristic roots of matrices II," Journal of Research of the National Bureau of Standards, 46(1951), 124-125.
- [10] O. Taussky-Todd, "Bibliography on Bounds for Characteristic Roots of Finite Matrices," National Bureau of Standards Report 1162, September 1951.
- [11] A. G. Walker and J. D. Weston, "Inclusion theorems for the eigenvalues of a normal matrix," Jour. London Math. Soc., 24(1949), 28-31.
- [12] H. Wielandt, "Ein Einschliessungssatz für charakteristische Wurzeln normaler Matrizen," Archiv der Math., 1(1948), 348-352.

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